# TOTAL DIFFERENTIAL EQUATIONS

## INTRODUCTION:

Total differential equations involve one independent variable (say $x$) and more than one dependent variables (say $y$ and $z$). In such equations the first order differential coefficient of these dependent variable with respect to the single independent variable (say $\frac{dy}{dx}, \frac{dz}{dx}$) will appear.

Obviously any such differential equation can be expressed in the form

$$Pdx + Qdy + Rdz = 0 \longrightarrow (1)$$

where $P, Q$ and $R$ are functions of $x, y$ and $z$.

This type of differential equation is said to be total differential equation or single differential equation or Pfaffian differential equation.

For example, the differential equation

$(zx^2 - y^3)dx + x^3y\,dy + x^2dz = 0$ is a total differential equation.

Also the differential equation,

$(x^2y+z)\frac{dy}{dx} + \sin yz \frac{dz}{dx} = e^{xyz}$ is a total differential equation, since this equation can be expressed as

$e^{xyz}dx - (x^2y+z)dy - \sin yz\,dz = 0$, which is of the form (1), where $P = e^{xyz}$, $Q = -(x^2y+z)$ and $R = -\sin yz$.

## CONDITION OF INTEGRABILITY:

The necessary and sufficient condition that a total differential equation $Pdx + Qdy + Rdz = 0$ has a solution $\phi(x,y,z) = \text{constant}$, is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

## METHOD-1: SOLUTION BY INSPECTION

Consider the equation $Pdx+Qdy+Rdz=0$, where P, Q and R are functions of $x$, $y$ and $z$.

In this method we convert the differential equation of the form $f_1(x)dx+f_2(y)dy+f_3(z)dz=0$ or we can express the differential equation in exact form s.t. we can easily integrate the differential equation.

**Example:→** Solve: $(yz+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$.

**Solution:→** Comparing the given equation to $Pdx+Qdy+Rdz=0$, we have

$P=yz+xyz$, $Q=zx+xyz$, $R=xy+xyz$.

| | | |
|---|---|---|
| $\frac{\partial P}{\partial x}=yz$ | $\frac{\partial Q}{\partial x}=z+yz$ | $\frac{\partial R}{\partial x}=y+yz$ |
| $\frac{\partial P}{\partial y}=z+xz$ | $\frac{\partial Q}{\partial y}=xz$ | $\frac{\partial R}{\partial y}=x+xz$ |
| $\frac{\partial P}{\partial z}=y+xy$ | $\frac{\partial Q}{\partial z}=x+xy$ | $\frac{\partial R}{\partial z}=xy$ |

Now $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$$=(yz+xyz)(x+xy-x-xz)+(zx+xyz)(y+yz-y-xy)$$
$$+(xy+xyz)(z+xz-z-yz)$$
$$=yz(1+x)x(y-z)+zx(1+y)y(z-x)+xy(1+z)z(x-y)$$
$$=xyz(1+x)(y-z)+xyz(1+y)(z-x)+xyz(1+z)(x-y)$$
$$=xyz(\cancel{y}-\cancel{z}+\cancel{xy}-\cancel{xz}+\cancel{z}-\cancel{x}+\cancel{yz}-\cancel{xy}+\cancel{x}-\cancel{y}+\cancel{xz}-\cancel{yz})$$
$$=0.$$

Hence the given differential equation is integrable.

Now $(yz+xyz)dx+(zx+xyz)dy+(xy+xyz)dz=0$

$\Rightarrow yz(1+x)dx+zx(1+y)dy+xy(1+z)dz=0$

$\Rightarrow \frac{1+x}{x}dx+\frac{1+y}{y}dy+\frac{1+z}{z}dz=0$

Integrating both sides we get,

$$\int\left(\frac{1}{x}+1\right)dx+\int\left(\frac{1}{y}+1\right)dy+\int\left(\frac{1}{z}+1\right)dz=0$$

$\Rightarrow \log x+x+\log y+y+\log z+z=c$, $c=$ arbitrary constant.

$\Rightarrow \log(xyz)+x+y+z=c$, which is the general solution.

## METHOD-II: ONE VARIABLE REGARDED AS CONSTANT

In this method one variable that does not maintain the similarity regarded as constant i.e. if we choose $x$ as constant then $dx=0$, therefore the equation reduces to $Qdy+Rdz=0$. Integrating this equation we get the required general solution.

**Examples** :→ Solve : $(2x^2+2xy+2xz^2+1)dx+dy+2zdz=0$.

**Solution** :→

$$(2x^2+2xy+2xz^2+1)dx+dy+2zdz=0 \longrightarrow (1)$$

Let, $x=$ constant.

Then $dx=0$.

Now the equation (1) reduces to $dy+2zdz=0$.

Integrating we get, $y+z^2=\phi$, where $\phi$ is independent of $y$ and $z$, i.e. may be a function of $x$.

$\therefore\ d\phi=dy+2zdz$.

Then the equation (1) reduces to

$d\phi+(2x^2+2x\phi+1)dx=0$. $\left[\because\ d\phi=dy+2zdz \text{ and } \phi=y+z^2.\right]$

$\Rightarrow \dfrac{d\phi}{dx}+2x^2+2x\phi+1=0$.

$\Rightarrow \dfrac{d\phi}{dx}+2x\phi=-(2x^2+1) \longrightarrow (2)$, which is a linear differential equation with dependent variable $\phi$ and independent variable $x$.

$\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

Multiplying both sides of (2) by $e^{x^2}$ we get,

$$e^{x^2}\frac{d\phi}{dx}+2x\cdot e^{x^2}\phi=-(2x^2+1)e^{x^2}$$

$$\Rightarrow d(e^{x^2}\cdot\phi)=-(2x^2+1)e^{x^2}\cdot dx$$

Integrating both sides we get,

$$e^{x^2}\phi=-\int(2x^2+1)e^{x^2}dx$$

$$=-\int 2x^2e^{x^2}dx-\int e^{x^2}dx$$

$$\therefore e^{x^2}\phi = -\int x\cdot 2xe^{x^2}dx - \int e^{x^2}dx$$

$$= -x\cdot\int 2xe^{x^2}dx + \int\left[\frac{d}{dx}(x)\int 2xe^{x^2}dx\right]dx - \int e^{x^2}dx.$$

$$= -x\cdot e^{x^2} + \int 1\cdot e^{x^2}dx - \int e^{x^2}dx$$

$$= -x\cdot e^{x^2} + c, \quad c = \text{arbitrary constant.}$$

$$\left[\begin{array}{l}\int 2x\cdot e^{x^2}dx \\ = \int e^{z}dz \\ = e^{z} \\ = e^{x^2}\end{array}\quad \begin{array}{l}\text{Let,} \\ x^2 = z \\ 2x\,dx = dz\end{array}\right]$$

$$\Rightarrow \phi = -x + ce^{-x^2}.$$

$$\therefore y + z^2 = -x + ce^{-x^2}.$$

$$\Rightarrow x + y + z^2 = ce^{-x^2},$$ which is the required general solution.

## METHOD-III : AUXILIARY EQUATIONS

Let, the given equation $Pdx + Qdy + Rdz = 0$ be integrable.

Then we have

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Comparing these two, we get,

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}.$$

These are called auxiliary equations and can be solved by methods discussed earlier.

Let, $u = a$ and $v = b$ be two integrals.

Now we wish to find A and B in such a way that the given equation can be written as

$$Adu + Bdv = 0.$$

So we find $Adu + Bdv = 0$ and compare it with $Pdx + Qdy + Rdz = 0$.

Then using $u = a$ and $v = b$, we obtain the values of A and B in terms of u and v. With these values of A and B in $Adu + Bdv =$ we get the required integral on integrati

**EXAMPLE :-** Solve: $z(z-y)dx+z(z+x)dy+x(x+y)dz=0$.

**SOLUTION:-**

$$z(z-y)dx+z(z+x)dy+x(x+y)dz=0 \longrightarrow (1)$$

Comparing the eqn (1) to the eqn $Pdx+Qdy+Rdz=0$ we get, $P=z(z-y)$, $Q=z(z+x)$ and $R=x(x+y)$.

$$\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}=2z+x-x=2z$$

$$\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=2x+y-2z+y=2(x+y-z).$$

$$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=-z-z=-2z.$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{2z}=\frac{dy}{2(x+y-z)}=\frac{dz}{-2z}.$$

Taking first and third term we have,

$$\int dx+\int dz=0.$$

$$\Rightarrow x+z=u, \text{ (say)}.$$

Also $$\frac{dx}{2z}=\frac{dy}{2(x+y-z)}=\frac{dz}{-2z}=\frac{dx+dy}{2(x+y)}.$$

Equating third and fourth term we get

$$\frac{dz}{z}+\frac{dx+dy}{(x+y)}=0$$

$$\Rightarrow \int\frac{dz}{z}+\int\frac{d(x+y)}{x+y}=0$$

$$\Rightarrow z(x+y)=v, \text{ (say)}$$

If the given eqn be identically equal to $Adu+Bdv=0$

i.e, $A(dx+dz)+B[z(dx+dy)+dz(x+y)]=0$.

or, $(A+Bz)dx+Bzdy+(A+Bx+By)dz=0$, then

$$A+Bz=z(z-y)$$

$$Bz=z(z+x)\Rightarrow B=z+x=u$$

$$\therefore A+z(z+x)=z(z-y)$$

$$\Rightarrow A=-(x+y)z=-v.$$

$\therefore$ The given eqn reduces to, $-vdu+udv=0$

$$\Rightarrow \frac{1}{u}du-\frac{1}{v}dv=0.$$

Integrating we get, $\frac{u}{v}=c$, $c=$ arbitrary constant.

i.e. $\frac{x+z}{(x+y)z}=c \Rightarrow x+z=cz(x+y)$, which is the required general solution.

## EXERCISES :→

1. Solve the following total differential equation:

i) $yz\,dx + 2zx\,dy - 3xy\,dz = 0$

ii) $yz(4xz+1)dx - zx(2xz+1)dy - xy\,dz = 0$

iii) $(x^2+y^2+z^2)\,dx - 2xy\,dy - 2xz\,dz = 0$

iv) $(a+y)^2 dx + z\,dy - (y+a)dz = 0$

v) $(zx^2 - y^3)dx + x^3 dz + 3xy^2 dy = 0$

vi) $y(yz+2x)\,dx - x^2 dy + y^2(2z+x)dz = 0$.

vii) $yz\log z\,dx + xy\,dz = xz\log z\,dy$.

viii) $z(x^2 - yz - z^2)dx + (x+z)xz\,dy + x(z^2 - x^2 + xy)dz = 0$

ix) $(y+z)dx + (x+z)dy + (x+y)dz = 0$

x) $(yz+xyz)dx + (zx+xyz)dy + (xy+xyz)dz = 0$

## ANSWERS :→

1.

i) $xy^2 - cz^3 = 0$

ii) $yz = cx(1+2xz)$

iii) $y^2+z^2-x^2 = cx$

iv) $z = (x+c)(y+a)$

v) $zx^2 + y^3 = cx$

vi) $x^2 + xyz + yz^2 = cy$

vii) $y = cx\log z$

viii) $\dfrac{x+y}{z} + \dfrac{y+z}{x} = c$

ix) $xy + yz + zx = c$

x) $\log xyz + (x+y+z) = c$

[**Note**: The type of answers may be different. For ex. the ans of 1.i) may be $z^3 - cxy^2 = 0$ ]

Prepared by
Pramita Bhar